

Sol Homework 4 Laplace Transforms : Operations b

1. 0/1 points

Find $F(s)$.

$$\mathcal{L}\{(t-1)u(t-1)\}$$

$$F(s) = \boxed{} \quad \times \quad \boxed{\frac{e^{-s}}{s^2}}$$

$$L\{f(t-a) \cdot u(t-a)\} = e^{-as} \cdot F(s) \quad \text{if} \quad L\{f(t)\} = F(s)$$

$$L\{(t-1) \cdot u(t-1)\} = e^{-(1) \cdot s} \cdot \frac{1!}{s^{1+1}} = \frac{e^{-s}}{s^2}$$

2. 0/1 points

Find $F(s)$.

$$\mathcal{L}\{(4t+1)u(t-1)\}$$

$$F(s) = \boxed{} \quad \times \quad \boxed{\frac{e^{-s}(5s+4)}{s^2}}$$

$$L\{f(t-a) \cdot u(t-a)\} = e^{-as} \cdot F(s) \quad \text{if} \quad L\{f(t)\} = F(s)$$

$$L\{(4t+1) \cdot u(t-1)\} = L\{(4t-4+4+1) \cdot u(t-1)\} = L\{[4(t-1)+5] \cdot u(t-1)\}$$

$$= L\{[4(t-1)] \cdot u(t-1)\} + L\{5 \cdot u(t-1)\} = 4 \cdot L\{[(t-1)] \cdot u(t-1)\} + 5 \cdot L\{u(t-1)\}$$

$$= 4 \cdot e^{-(1) \cdot s} \cdot \frac{1!}{s^{1+1}} + 5 \cdot \frac{e^{-(1) \cdot s}}{s} = \frac{4e^{-s}}{s^2} + \frac{5e^{-s}}{s} = \frac{4e^{-s} + 5se^{-s}}{s^2} = \frac{e^{-s}(5s+4)}{s^2}$$

3. 0/3 points

Use the Laplace transform to solve the given initial-value problem.

$$y' + 6y = f(t), \quad y(0) = 0, \quad \text{where } f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 0, & t \geq 1 \end{cases}$$

$$y(t) = \boxed{} \times \boxed{\frac{1}{36} (6t + e^{-6t} - 1)} + \left(\boxed{} \times \boxed{\frac{1}{36} (-6t + 5e^{6(1-t)} + 1)} \right) \mathcal{U}\left(t - \boxed{} \times \boxed{1}\right)$$

$$f(t) = t \cdot u(t) - t \cdot u(t-1) = t \cdot u(t) - (t-1) \cdot u(t-1) - u(t-1)$$

$$y' + 6y = f(t) \quad y(0) = 0$$

$$y' + 6y = t \cdot u(t) - (t-1) \cdot u(t-1) - u(t-1) \quad y(0) = 0$$

$$L\{y' + 6y\} = L\{t \cdot u(t) - (t-1) \cdot u(t-1) - u(t-1)\} = L\{(t-0) \cdot u(t-0) - (t-1) \cdot u(t-1) - u(t-1)\}$$

$$sY(s) - y(0) + 6Y(s) = \frac{1!}{s^{1+1}} e^{-(0)s} - \frac{1!}{s^{1+1}} e^{-(1)s} - \frac{e^{-s}}{s}$$

$$sY(s) - 0 + 6Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

$$(s+6)Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

$$Y(s) = \frac{1}{s^2(s+6)} - \frac{e^{-s}}{s^2(s+6)} - \frac{e^{-s}}{s(s+6)}$$

$$Y(s) = \frac{1}{s^2(s+6)} - \frac{1}{s^2(s+6)} e^{-s} - \frac{1}{s(s+6)} e^{-s}$$

$$Y(s) = \left[\frac{\left(\frac{1}{36}\right)}{s+6} + \frac{\left(-\frac{1}{36}\right)}{s} + \frac{\left(\frac{1}{6}\right)}{s^2} \right] - \left[\frac{\left(\frac{1}{36}\right)}{s+6} + \frac{\left(-\frac{1}{36}\right)}{s} + \frac{\left(\frac{1}{6}\right)}{s^2} \right] \cdot e^{-s} - \left[\frac{\left(-\frac{1}{6}\right)}{s+6} + \frac{\left(\frac{1}{6}\right)}{s} \right] \cdot e^{-s}$$

$$Y(s) = \frac{\left(\frac{1}{36}\right)}{s+6} + \frac{\left(-\frac{1}{36}\right)}{s} + \frac{\left(\frac{1}{6}\right)}{s^2} - \frac{\left(\frac{1}{36}\right)}{s+6}e^{-s} - \frac{\left(-\frac{1}{36}\right)}{s}e^{-s} - \frac{\left(\frac{1}{6}\right)}{s^2}e^{-s} - \frac{\left(-\frac{1}{6}\right)}{s+6}e^{-s} - \frac{\left(\frac{1}{6}\right)}{s}e^{-s}$$

$$Y(s) = \frac{\left(\frac{1}{36}\right)}{s+6} + \frac{\left(-\frac{1}{36}\right)}{s} + \frac{\left(\frac{1}{6}\right)}{s^2} + \frac{\left(\frac{5}{36}\right)}{s+6}e^{-s} - \frac{\left(\frac{5}{36}\right)}{s}e^{-s} - \frac{\left(\frac{1}{6}\right)}{s^2}e^{-s}$$

$$y(t) = \frac{1}{36}e^{-6t} - \frac{1}{36} + \frac{1}{6}t + \frac{5}{36}e^{-6(t-1)} \cdot u(t-1) - \frac{5}{36}u(t-1) - \frac{1}{6}(t-1) \cdot u(t-1)$$

$$y(t) = \frac{1}{36}e^{-6t} - \frac{1}{36} + \frac{6}{36}t + \frac{5}{36}e^{-6(t-1)} \cdot u(t-1) - \frac{5}{36}u(t-1) - \frac{6}{36}(t-1) \cdot u(t-1)$$

$$y(t) = \frac{1}{36}(6t + e^{-6t} - 1) + \frac{5}{36}e^{-6(t-1)}u(t-1) - \frac{5}{36}u(t-1) - \frac{6}{36}t \cdot u(t-1) + \frac{6}{36}u(t-1)$$

$$y(t) = \frac{1}{36}(6t + e^{-6t} - 1) + \frac{5}{36}e^{-6(t-1)} \cdot u(t-1) + \frac{1}{36}u(t-1) - \frac{6}{36}t \cdot u(t-1)$$

$$y(t) = \frac{1}{36}(6t + e^{-6t} - 1) + \frac{1}{36}[-6t + 5e^{-6(t-1)} + 1]u(t-1)$$

$$y(t) = \frac{1}{36}(6t + e^{-6t} - 1) + \frac{1}{36}[-6t + 5e^{6(1-t)} + 1]u(t-1)$$

4. + 0/1 points

Use Theorem 7.4.1.

THEOREM 7.4.1 Derivatives of Transforms

If $F(s) = \mathcal{L}\{f(t)\}$ and $n = 1, 2, 3, \dots$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

Evaluate the given Laplace transform. (Write your answer as a function of s .)

$$\mathcal{L}\{te^{-6t}\}$$

$\times \frac{1}{(s+6)^2}$

$$L\{te^{-6t}\} = (-1) \cdot \frac{d}{ds} \left(L\{e^{-6t}\} \right)$$

$$L\{te^{-6t}\} = -\frac{d}{ds} \left(\frac{1}{s+6} \right) = \frac{1}{(s+6)^2}$$

5. + 0/1 points

Use Theorem 7.4.1.

THEOREM 7.4.1 Derivatives of Transforms

If $F(s) = \mathcal{L}\{f(t)\}$ and $n = 1, 2, 3, \dots$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

Evaluate the given Laplace transform. (Write your answer as a function of s .)

$$\mathcal{L}\{t \cos(5t)\}$$

$\times \frac{s^2 - 25}{(s^2 + 25)^2}$

$$L\{t \cos(5t)\} = (-1) \frac{d}{ds} \left(L\{\cos(5t)\} \right) = -\frac{d}{ds} \left(\frac{s}{s^2 + 5^2} \right) = -\frac{(s^2 + 5^2)(1) - s(2s)}{(s^2 + 5^2)^2}$$

$$L\{t \cos(5t)\} = \frac{s^2 - 25}{(s^2 + 25)^2}$$

6. 0/1 points

Use the Laplace transform to solve the given initial-value problem. Use the [table of Laplace transforms in Appendix III](#) as needed.

$$y' + y = t \sin t, \quad y(0) = 0$$

$$y(t) = \boxed{} \quad \times \quad \boxed{-\frac{e^{-t}}{2} + \frac{1}{2}t \sin(t) - \frac{1}{2}t \cos(t) + \frac{\cos(t)}{2}}$$

$$y' + y = t \cdot \sin t \quad y(0) = 0$$

$$L\{y' + y\} = L\{t \cdot \sin t\}$$

$$sY(s) - y(0) + Y(s) = (-1) \cdot \frac{d}{ds} \left(\frac{1}{s^2 + 1} \right)$$

$$sY(s) - 0 + Y(s) = -\frac{(-2s)}{(s^2 + 1)^2}$$

$$sY(s) + Y(s) = \frac{2s}{(s^2 + 1)^2}$$

$$(s + 1)Y(s) = \frac{2s}{(s^2 + 1)^2}$$

$$Y(s) = \frac{2s}{(s^2 + 1)^2 (s + 1)}$$

Con la ayuda de Matlab:

Command Window

```
>> format rat
>> syms s
>> F=2*s/((s+1)*(s^2+1)^2);
>> partfrac(F,s)

ans =

(s/2 - 1/2)/(s^2 + 1) - 1/(2*(s + 1)) + (s + 1)/(s^2 + 1)^2
```

Es decir:

$$Y(s) = \frac{\frac{s}{2} - \frac{1}{2}}{s^2 + 1} - \frac{1}{2(s+1)} + \frac{s+1}{(s^2+1)^2} = -\frac{1}{2} \frac{1}{(s+1)} + \frac{1}{2} \frac{(s-1)}{(s^2+1)} + \frac{s+1}{(s^2+1)^2}$$

En lugar de usar las tablas, como el problema sugiere, usamos Matlab con el comando ***partfrac()*** en lugar del comando *residue()* en virtud de que existen factores cuadráticos irreducibles a factores lineales con números reales:

 Command Window

```
>> format rat
>> syms s
>> F=2*s/((s+1)*(s^2+1)^2);
>> partfrac(F,s)

ans =

(s/2 - 1/2)/(s^2 + 1) - 1/(2*(s + 1)) + (s + 1)/(s^2 + 1)^2

>> ilaplace(ans)

ans =

cos(t)/2 - exp(-t)/2 - (t*cos(t))/2 + (t*sin(t))/2
```

Reacomodando la respuesta de Matlab:

$$y(t) = -\frac{e^{-t}}{2} + \frac{1}{2}t \sin(t) - \frac{1}{2}t \cos(t) + \frac{\cos(t)}{2}$$

7. 0/1 points

Use the Laplace transform to solve the given initial-value problem. Use the [table of Laplace transforms in Appendix III](#) as needed.

$$y'' + 16y = \cos(4t), \quad y(0) = 3, \quad y'(0) = 2$$

$$y(t) = \boxed{} \times \boxed{\frac{1}{8}t \sin(4t) + \frac{1}{2} \sin(4t) + 3 \cos(4t)}$$

$$y'' + 16y = \cos(4t) \quad y(0) = 3 \quad y'(0) = 2$$

$$L\{y'' + 16y\} = L\{\cos(4t)\}$$

$$s^2 Y(s) - s \cdot y(0) - y'(0) + 16Y(s) = \frac{s}{s^2 + 16}$$

$$s^2 Y(s) - 3s - 2 + 16Y(s) = \frac{s}{s^2 + 16}$$

$$s^2 Y(s) + 16Y(s) = \frac{s}{s^2 + 16} + 3s + 2$$

$$(s^2 + 16)Y(s) = \frac{s}{s^2 + 16} + 3s + 2$$

$$Y(s) = \frac{s}{(s^2 + 16)^2} + \frac{3s + 2}{s^2 + 16}$$

Utilizando Matlab:

 Command Window

```
>> format rat
>> syms s
>> F=s/(s^2+16)^2+(3*s+2)/(s^2+16);
>> ilaplace(F)

ans =

3*cos(4*t) + sin(4*t)/2 + (t*sin(4*t))/8
```

Reacomodando el resultado de Matlab:

$$y(t) = \frac{1}{8}t \sin(4t) + \frac{1}{2} \sin(4t) + 3 \cos(4t)$$

8. 0/3 points

Use the Laplace transform to solve the given initial-value problem. Use the [table of Laplace transforms in Appendix III](#) as needed.

$$y'' + 9y = f(t), \quad y(0) = 0, \quad y'(0) = 1, \text{ where}$$

$$f(t) = \begin{cases} \cos 3t, & 0 \leq t < \pi \\ 0, & t \geq \pi \end{cases}$$

$$y(t) = \boxed{} \times \left[\frac{1}{6}t \sin(3t) + \frac{1}{3} \sin(3t) \right] + \left(\boxed{} \times \left[\frac{1}{6}(t - \pi) \sin(3(t - \pi)) \right] \right) u(t - \boxed{} \times \boxed{\pi})$$

$$y'' + 9y = f(t) \quad y(0) = 0 \quad y'(0) = 1$$

$$f(t) = \cos(3t) \cdot u(t) - \cos(3t) \cdot u(t - \pi)$$

$$L\{y'' + 9y\} = L\{\cos(3t) \cdot u(t) - \cos(3t) \cdot u(t - \pi)\}$$

$$L\{y'' + 9y\} = L\{\cos(3t) \cdot u(t) - \cos(3t - 2\pi) \cdot u(t - \pi)\}$$

$$L\{y'' + 9y\} = L\{\cos(3t) \cdot u(t) + \cos[\pi - (3t - 2\pi)] \cdot u(t - \pi)\}$$

$$L\{y'' + 9y\} = L\{\cos(3t) \cdot u(t) + \cos(3\pi - 3t) \cdot u(t - \pi)\}$$

$$L\{y'' + 9y\} = L\{\cos(3t) \cdot u(t) + \cos(3t - 3\pi) \cdot u(t - \pi)\}$$

$$L\{y'' + 9y\} = L\{\cos(3t) \cdot u(t) + \cos[3(t - \pi)] \cdot u(t - \pi)\}$$

$$s^2 Y(s) - s \cdot y(0) - y'(0) + 9Y(s) = \frac{s}{s^2 + 3^2} + \frac{s}{s^2 + 3^2} e^{-\pi s}$$

$$s^2 Y(s) - s \cdot 0 - 1 + 9Y(s) = \frac{s}{s^2 + 3^2} + \frac{s}{s^2 + 3^2} e^{-\pi s}$$

$$s^2 Y(s) + 9Y(s) = \frac{s}{s^2 + 9} + \frac{s}{s^2 + 9} e^{-\pi s} + 1$$

$$(s^2 + 9)Y(s) = \frac{s}{s^2 + 9} + \frac{s}{s^2 + 9} e^{-\pi s} + 1$$

$$Y(s) = \frac{s}{(s^2 + 9)^2} + \frac{s}{(s^2 + 9)^2} e^{-\pi s} + \frac{1}{s^2 + 9}$$

Utilizando Matlab:

Command Window

```
>> format rat
>> syms s
>> F=s/(s^2+9)^2+s*exp(-pi*s)/(s^2+9)+1/(s^2+9);
>> ilaplace(F)

ans =

sin(3*t)/3 + (t*sin(3*t))/6 - (sin(3*t)*heaviside(t - pi)*(t - pi))/6
```

Es decir:

$$y(t) = \frac{1}{6}t \sin(3t) + \frac{1}{3}\sin(3t) - \frac{1}{6}(t - \pi) \cdot \sin(3t) \cdot u(t - \pi)$$

o, de manera equivalente (sólo para hacer coincidir el resultado con el de Webassign):

$$y(t) = \frac{1}{6}t \sin(3t) + \frac{1}{3}\sin(3t) - \frac{1}{6}(t - \pi) \cdot \sin(3t - 2\pi) \cdot u(t - \pi)$$

$$y(t) = \frac{1}{6}t \sin(3t) + \frac{1}{3}\sin(3t) - \frac{1}{6}(t - \pi) \cdot \sin[\pi - (3t - 2\pi)] \cdot u(t - \pi)$$

$$y(t) = \frac{1}{6}t \sin(3t) + \frac{1}{3}\sin(3t) - \frac{1}{6}(t - \pi) \cdot \sin(3\pi - 3t) \cdot u(t - \pi)$$

$$y(t) = \frac{1}{6}t \sin(3t) + \frac{1}{3}\sin(3t) + \frac{1}{6}(t - \pi) \cdot \sin(3t - 3\pi) \cdot u(t - \pi)$$

$$y(t) = \frac{1}{6}t \sin(3t) + \frac{1}{3}\sin(3t) + \frac{1}{6}(t - \pi) \cdot \sin[3(t - \pi)] \cdot u(t - \pi)$$

9. + 0/1 points

Use (8), $\int_0^t f(\tau) d\tau = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}$, to evaluate the given inverse transform. (Write your answer as a function of t .)

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\}$$

✗ $e^t - 1$

$$\begin{aligned} L^{-1}\left\{\frac{1}{s(s-1)}\right\} &= L^{-1}\left\{\frac{1}{s-1} \cdot \frac{1}{s}\right\} \\ &= \int_0^t e^t dt = e^t \Big|_0^t = e^t - e^0 = e^t - 1 \end{aligned}$$

10. + 0/1 points

Use (8), $\int_0^t f(\tau) d\tau = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}$, to evaluate the given inverse transform. (Write your answer as a function of t .)

$$\mathcal{L}^{-1}\left\{\frac{7}{s^2(s-1)}\right\}$$

✗ $-7t + 7e^t - 7$

$$L^{-1}\left\{\frac{7}{s^2(s-1)}\right\} = L^{-1}\left\{\frac{7}{s(s-1)} \cdot \frac{1}{s}\right\} = 7 \cdot L^{-1}\left\{\frac{1}{s(s-1)}\right\}$$

Considerando el resultado del problema 9 anterior:

$$L^{-1}\left\{\frac{1}{s(s-1)}\right\} = e^t - 1$$

Nuestros cálculos se transforman de la siguiente manera:

$$\begin{aligned} L^{-1}\left\{\frac{7}{s^2(s-1)}\right\} &= L^{-1}\left\{\frac{\frac{7}{s(s-1)}}{s}\right\} = 7 \cdot L^{-1}\left\{\frac{\frac{1}{s(s-1)}}{s}\right\} = 7 \cdot \int_0^t (e^t - 1) dt \\ &= 7 \cdot (e^t - t) \Big|_0^t = 7 \cdot [(e^t - t) - (e^0 - 0)] = 7 \cdot (e^t - t - 1) = -7t + 7e^t - 7 \end{aligned}$$